VI. *On the Exact Form of Waves near the Surface of Deep Water.*

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(1.) The investigations of the Astronomer Royal and of some other mathematicians on straight-crested parallel waves in a liquid, are based on the supposition that the displacements of the particles of the liquid are small compared with the length of a wave. Hence it has been very generally inferred that the results of those investigations are approximate only, when applied to waves in which the displacements, as compared with the length of a wave, are considerable.

(2.) In the present paper I propose to prove that one of those results (viz., that in very deep water the particles move with a uniform angular velocity in vertical circles whose radii diminish in geometrical progression with increased depth, and consequently that surfaces of equal pressure, including the upper surface, are trochoidal) is exact for all displacements, how great soever.

(3.) I believe the trochoidal form of waves to have been first explicitly stated by Mr. Scott Russell; but no demonstration of its exactly fulfilling the conditions of the question has yet been published, so far as I know.

(4.) In 'A Manual of Applied Mechanics' (first published in 1858), page 579, I stated that the theory of rolling waves might be deduced from that of the positions assumed by the surface of a mass of water revolving in a vertical plane about a horizontal axis; as the theory of such waves, however, was foreign to the subject of the book, I did not then publish the investigation on which that statement was founded.

(5.) Having communicated some of the leading principles of that investigation to Mr. William Froude in April 1862, I learned from him that he had already arrived independently at similar results by a similar process, although he had not published them.

(6.) Proposition I.—*In a mass of gravitating liquid whose particles revolve uniformly in vertical circles, a wavy surface of trochoidal profile fulfils the conditions of uniformity of pressure,—such trochoidal profile being generated by rolling, on the underside of a straight line, a circle whose radius is equal to the height of a conical pendulum that revolves in the same period with the particles of liquid.*

In fig. 1 (p. 128) let B be a particle of liquid revolving uniformly in a vertical circle of the radius C B, in the direction indicated by the arrow N; and let it make n revolutions in a second. Then the centrifugal force of B (taking its mass as unity) will be

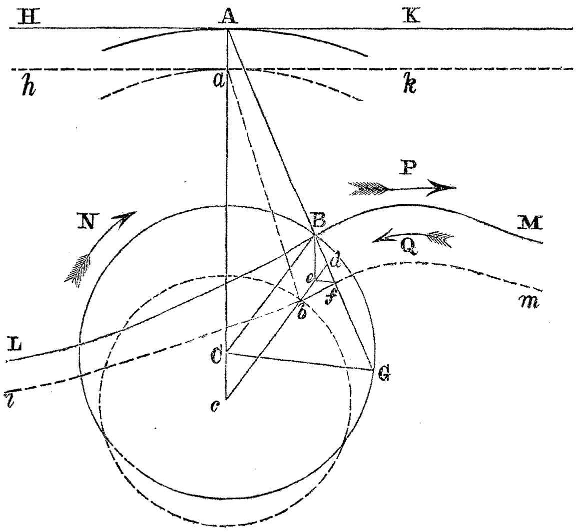
$$4\pi^2 n^2 \cdot CB.$$

Draw $\overline{CA}$ vertically upwards, and of such a length that centrifugal force : gravity :: $\overline{CB}:\overline{AC}$; that is to say, make

$$\overline{AC}=\frac{g}{4\pi^2n^2};$$

which is the well-known expression for the height of a revolving pendulum making n revolutions in a second.

Fig. 1.



Then $\overline{AC}$ being in the direction of and proportional to gravity, and $\overline{CB}$ in the direction of and proportional to centrifugal force, $\overline{AB}$ will be in the direction of and proportional to the resultant of gravity and centrifugal force; and the surface of equal pressure traversing B will be normal to $\overline{AB}$.

The profile of such a surface is obviously a trochoid LBM, traced by the point B, which is carried by a circle of the radius $\overline{CA}$ rolling along the underside of the horizontal straight line HAK. Q.E.D.

(7.) *Corollaries.*—The length of the wave whose period is one-nth of a second is equal to the circumference of the rolling circle; that is to say (denoting that length by λ),

$$\lambda=2\pi.\overline{CA}=\frac{g}{2\pi n^2}:$$

the period of a wave of a given length λ is given in seconds, or fractions of a second, by the equation

$$\frac{1}{n}=\sqrt{\frac{2\pi\lambda}{g}};$$

and the velocity of propagation of such a wave is

$$n\lambda=\frac{g}{2\pi n}=\sqrt{\frac{g\lambda}{2\pi}}=\sqrt{g.\overline{CA}};$$

results agreeing with those of the known theory.

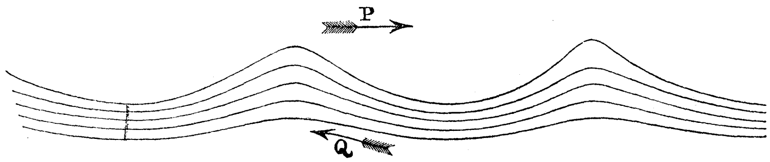
(8.) PROPOSITION II.—*Let another surface of uniform pressure be conceived to exist indefinitely near to the first surface; then, if the first surface is a surface of continuity, so also is the second.*

By a surface of continuity is here meant one which always passes through the same set of particles of liquid, so that a pair of such surfaces contain between them a layer of particles which are always the same.

The perpendicular distance between a pair of surfaces of uniform pressure is in this case inversely proportional to the resultant of gravity and centrifugal force; that is to say, to the normal $\overline{AB}$. Hence if a curve *lbfm* be drawn indefinitely near to the curve LBM, so that the perpendicular distance between them, $\overline{Bf}$, shall everywhere be inversely proportional to the normal $\overline{AB}$, the second curve will also be the profile of a surface of uniform pressure.

Conceive now that the whole mass of liquid has, combined with its wave-motion, a uniform motion of translation, with a velocity equal and opposite to that of the propagation of the waves. The dynamical conditions of the mass are not in the least altered by this; but the forms of the waves are rendered stationary (as we sometimes see in a rapid stream), and, instead of a series of waves propagated in the direction shown by the arrow P, we have an *undulating current* running the reverse way, in the direction shown by the arrow Q. (This is further illustrated by fig. 2.) According to a well-known

Fig. 2.



property of curves described by rolling, the velocity of the particle B in that current is proportional to the normal $\overline{AB}$, and is given by the expression

$$2\pi n . \overline{AB}.$$

Consider the layer of the current contained between the surfaces LBM and *lbm*. In order that the latter of those surfaces, as well as the former, may be a surface of continuity, it is necessary and sufficient that the thickness of the layer $\overline{Bf}$ at each point should be inversely as the velocity; and that condition is already fulfilled; for $\overline{Bf}$ varies inversely as $\overline{AB}$, and $\overline{AB}$ varies as the velocity of the current at B; therefore LBM and *lbm* are not only a pair of surfaces of uniform pressure, but a pair of surfaces of continuity also. Q.E.D.

(9.) *Corollary.*—The surfaces of uniform pressure are identical with surfaces of continuity throughout the whole mass of liquid.

(10.) *Corollary.*—Inasmuch as the resultant of gravity and centrifugal force at B is represented by

$$g \cdot \frac{\overline{AB}}{\overline{AC}},$$

the excess of the uniform pressure at the surface *lbm* above that at the surface LBM is given by the expression

$$dp = w \cdot \frac{\overline{\mathrm{AB}}}{\overline{\mathrm{AC}}} \cdot \overline{\mathrm{B}f},$$

in which w is the heaviness of the liquid, in units of weight per unit of volume. By omitting the factor w, the pressure is expressed in units of height of a column of the liquid.

(11.) PROPOSITION III.—*The profile of the lower surface of the layer referred to in the preceding proposition is a trochoid generated by a rolling circle of the same radius with that which generates the first trochoid; and the tracing-arm of the second trochoid is shorter than that of the first trochoid by a quantity bearing the same proportion to the depth of the centre of the second rolling circle below the centre of the first rolling circle, which the tracing-arm of the first rolling circle bears to the radius of that circle.*

At an indefinitely small depth $\overline{\mathrm{A}a}$ below the horizontal line HAK, draw a second horizontal line *hak*, on the under side of which let a circle roll with a radius $\overline{ca} = \overline{\mathrm{CA}}$, the radius of the first rolling circle; so that the indefinitely small depths $\overline{\mathrm{C}c} = \overline{\mathrm{A}a}$. To find the tracing-arm of the second rolling circle, draw cd parallel to $\overline{\mathrm{CB}}$, the tracing-arm of the first circle; in cd take $\overline{ce} = \overline{\mathrm{CB}}$, and cut off $\overline{eb} = \overline{ed}$; b will be the tracing-point, and $\overline{cb}$ the tracing-arm required; for, according to the principle laid down in the enunciation, we are to have

$$\overline{\mathrm{CB}} - \overline{cb} = \overline{eb} = \mathrm{C}c \cdot \frac{\overline{\mathrm{CB}}}{\overline{\mathrm{CA}}}.$$

Let the second circle roll; then b will trace a trochoid *lbm*. From b let fall bf perpendicular to $\overline{\mathrm{AB}}$ produced; Bf will be the indefinitely small thickness at B of the layer between the two trochoidal surfaces.

The proposition enunciated amounts to stating that Bf is everywhere inversely proportional to the normal $\overline{\mathrm{AB}}$; so that *lbm* is the profile of a surface of uniform pressure and of continuity.

To prove this, join $\overline{\mathrm{B}e}$ and $\overline{ef}$. Then Be is parallel to ACc, and equal to Cc, and def is evidently an isosceles triangle, $\overline{ef}$ being $= ed$. Let AB (produced if necessary) cut the circle of the radius CB in G; then CG is parallel to ef, and the indefinitely small triangle Bef is similar to the triangle ACG; consequently $\overline{\mathrm{AC}} : \overline{\mathrm{AG}} :: \overline{\mathrm{B}e} = \overline{\mathrm{C}c} : \mathrm{B}f$; or

$$\overline{\mathrm{B}f} = \overline{\mathrm{C}c} \cdot \frac{\overline{\mathrm{AG}}}{\overline{\mathrm{AC}}};$$

but, by a well-known property of the circle,

$$\overline{\mathrm{AG}} = \frac{\overline{\mathrm{AC}}^2 - \overline{\mathrm{CB}}^2}{\overline{\mathrm{AB}}};$$

and therefore

$$\overline{\mathrm{B}f} = \overline{\mathrm{C}c} \cdot \frac{\overline{\mathrm{AC}}^2 - \overline{\mathrm{CB}}^2}{\overline{\mathrm{AC}} \cdot \overline{\mathrm{AB}}};$$

that is to say, *the thickness of the layer varies inversely as the normal* $\overline{AB}$; *and the second trochoid, lbm, is therefore the profile of a surface of uniform pressure and of continuity.* Q. E. D.

(12.) *Corollaries.*—The profiles of the surfaces of uniform pressure and of continuity form an indefinite series of trochoids, described by equal rolling circles, rolling with the same speed below an indefinite series of horizontal straight lines.

The tracing-arms of those circles (each of which arms is the radius of the circular orbit of the particles contained in the trochoidal surface which it traces) diminish in geometrical progression with increase of depth, according to the following laws:—

For convenience, let $\overline{Cc}$ be denoted by dk, $\overline{CB}$ by r, and $\overline{cb}$ by $r-dr$; then

$$dr=dk.\frac{r}{\overline{AC}}=dk.\frac{r}{2\pi\lambda},$$

and the integration of this equation gives the following result:—

Let k denote the vertical depth of the centre of the generating circle of a given surface below the centre of the generating circle of the free upper surface of the liquid;

r_0 the tracing-arm of the free upper surface (= half the amplitude of disturbance);

r_1 the tracing-arm of the surface whose middle depth is k; then

$$r_1=r_0e^{-\frac{k}{\overline{AC}}}=r_0e^{-\frac{2\pi k}{\lambda}},$$

a formula exactly agreeing with that found for indefinitely small disturbances by previous investigators.

(13.) PROPOSITION IV.—*The centres of the orbits of the particles in a given surface of equal pressure stand at a higher level than the same particles do when the liquid is still, by a height which is a third proportional to the diameter of the rolling circle and the tracing-arm or radius of the orbits of the particles, and which is equal to the height due to the velocity of revolution of the particles.*

If the liquid were still, the given surface of equal pressure would become horizontal. To find the level at which it would stand, we must first find what relation the mean vertical depth of a given layer of particles bears to the depth $\overline{Cc}=dk$, between the centres of the rolling circles that generate its boundaries.

The length of the arc of the curve LBM described in an indefinitely short interval of time dt is

$$2\pi n.\overline{AB}.dt,$$

and the thickness of the layer being

$$\overline{Bf}=dk.\frac{\overline{AC}^2-\overline{CB}^2}{\overline{AC}.\overline{AB}},$$

let the product of those quantities be divided by the distance through which the centre of the rolling circle moves in the same time, viz.

$$2\pi n.\overline{AC}.dt,$$

and the result will be the mean vertical depth of the layer, which being denoted by dk_0 we have

$$dk_0 = dk . \left(1 - \frac{\overline{CB}^2}{\overline{AC}^2}\right) = dk . \left(1 - \frac{r^2}{\overline{AC}^2}\right) = dk . \left(1 - \frac{r_0^2}{\overline{AC}^2} e^{-\frac{2k}{AC}}\right).$$

The difference by which the mean vertical thickness of the layer falls short of the difference of level of the rolling circles of its upper and lower surfaces is given by the following expression,

$$dk - dk_0 = \frac{r_0^2}{\overline{AC}^2} e^{-\frac{2k}{AC}} dk ;$$

and this being integrated from ∞ to k, gives the depth of the position of a given particle, when the liquid is still, below the level of the centre of the orbit of the same particle when disturbed, viz.

$$k_0 - k = \frac{r_0^2}{2\overline{AC}} . e^{-\frac{2k}{AC}} = \frac{r^2}{2\overline{AC}} = \frac{\pi r^2}{\lambda},$$

or *a third proportional to the diameter of the rolling circle and the radius of the orbit of the particle; also* $\frac{r^2}{2\overline{AC}} = \frac{4\pi^2 n^2 r^2}{2g}$ *is the height due to the velocity of revolution of the particles.* Q.E.D.

(13 A.) *Corollary.*—The mechanical energy of a wave is half actual and half potential,—half being due to motion, and half to elevation. In other words, the mechanical energy of a wave is double of that due to the motion of its particles only, there being an equal amount due to the mean elevation of the particles above their position when the water is still.

(14.) *Corollary.*—The crests of the waves rise higher above the level of still water than their hollows fall below it; and the difference between the elevation of the crest and the depression of the hollow is double of the quantity mentioned in Proposition IV., that is to say, it is

$$\frac{r^2}{\overline{AC}} = \frac{2\pi r^2}{\lambda}.$$

(15.) *Corollary as to Pressures.*—An expression has already been given in art. 10 for the difference of pressure at the upper and under surfaces of a given layer. Substituting in that expression the value of the thickness of the layer, we find

$$d\bar{p} = w . \frac{\overline{AB}}{\overline{AC}} . dk . \frac{\overline{AC}^2 - \overline{CB}^2}{\overline{AC} . \overline{AB}} = w . dk \left(1 - \frac{\overline{CB}^2}{\overline{AC}^2}\right) = w . dk_0$$

(as the preceding corollary shows), being precisely the same as if the liquid were still; and hence it follows that *the hydrostatic pressure at each individual particle during wave-motion is the same as if the liquid were still.*

(16.) In Proposition III. it has been shown, by geometrical reasoning from the mechanical construction of the trochoid, that a wave consisting of trochoidal layers satisfies the condition of continuity. It may be satisfactory also to show the same thing by the use of algebraic symbols. For that purpose the following notation will be used.

Let the origin of coordinates be assumed to be in the horizontal line containing the centre of the circle which is rolled to trace the profile of *cycloidal* waves, having cusps, and being (as Mr. SCOTT RUSSELL long ago pointed out) the highest waves that can exist without breaking. In such waves, the tracing-arm, or radius vector, of the uppermost particles is equal to the radius of the rolling circle; and that arm diminishes for each successive layer proceeding downwards.

Let x and y be the coordinates of any particle, x being measured horizontally *against* the direction of propagation, and y vertically downwards.

Let k (as before) be the vertical coordinate of the centre of the given particle's orbit; h the horizontal coordinate of the same centre.

Let R be the radius of the rolling circle, a the angular velocity of the tracing-arm $(=2\pi n)$, so that

$$2\pi\mathrm{R}=\lambda$$

is the length of a wave, and

$$a\mathrm{R}=n\lambda\sqrt{\frac{g\lambda}{2\pi}}=\sqrt{g\mathrm{R}}$$

is the velocity of propagation.

Let θ denote the *phase* of the wave at a given particle, being the angle which its radius vector, or tracing-arm, makes with the direction of $+y$, that is, with a line pointing vertically downwards.

Let t denote time, reckoned from the instant at which all the particles for which $h=0$ are in the axis of y; then

$$\theta=at+\frac{h}{\mathrm{R}}. \qquad (1.)$$

Then the following equations give the coordinates of a given particle at a given instant:

$$x=h+\mathrm{R}e^{-\frac{k}{\mathrm{R}}}\sin\theta; \qquad (2.)$$

$$y=k+\mathrm{R}e^{-\frac{k}{\mathrm{R}}}\cos\theta. \qquad (3.)$$

Let u and v denote the vertical and horizontal components of the velocity of the particle at the given instant; then

$$u=\frac{dx}{dt}=\ a\mathrm{R}\,.\,e^{-\frac{k}{\mathrm{R}}}\cos\theta=\ a(y-k); \qquad (4.)$$

$$v=\frac{dy}{dt}=-a\mathrm{R}\,.\,e^{-\frac{k}{\mathrm{R}}}\sin\theta=-a(x-h). \qquad (5.)$$

The well-known equation of continuity in a liquid in two dimensions is

$$\frac{du}{dx}+\frac{dv}{dy}=0; \qquad (6.)$$

and from equations (4.) and (5.) it appears that we have in the present case

$$\frac{du}{dx}+\frac{dv}{dy}=a\left(-\frac{dk}{dx}+\frac{dh}{dy}\right)=a\left(-\frac{dk}{dx}+\frac{\mathrm{R}d\theta}{dy}\right). \qquad (7.)$$

In the original formulæ, k and θ are the independent variables. When x and y are made the independent variables instead, we have, by well-known formulæ,

$$\left.\begin{aligned} \frac{dk}{dx}&=1\div\left\{\frac{dx}{dk}-\frac{dx}{d\theta}\cdot\frac{\frac{dy}{dk}}{\frac{dy}{d\theta}}\right\}=\frac{e^{-\frac{k}{R}}\sin\theta}{1-e^{-\frac{2k}{R}}} \\ \text{and} \\ \frac{d\theta}{dy}&=1\div\left\{\frac{dy}{d\theta}-\frac{dy}{dk}\cdot\frac{\frac{dx}{d\theta}}{\frac{dx}{dk}}\right\}=\frac{e^{-\frac{k}{R}}\sin\theta}{R\left(1-e^{-\frac{2k}{R}}\right)} \end{aligned}\right\}, \quad \ldots \ldots \quad (8.)$$

so that the equation of continuity (6.) is exactly verified.

(17.) Another mode of testing algebraically the fulfilment of the condition of continuity is the following. It is analogous to that employed by Mr. AIRY; but inasmuch as the disturbances in the present paper are regarded as considerable compared with the length of a wave, it takes into account quantities which, in Mr. AIRY's investigation, are treated as inappreciable.

Consider an indefinitely small rhomboidal particle, bounded by surfaces for which the values of h and k are respectively h, $h+dh$, k, $k+dk$. Then the area of that rhomboid is

$$\left(\frac{dx}{dh}\cdot\frac{dy}{dk}-\frac{dx}{dk}\cdot\frac{dy}{dh}\right)dh\,.\,dk;$$

and the condition of continuity is that this area shall be at all times the same; that is to say, that

$$\frac{d}{dt}\left(\frac{dx}{dh}\cdot\frac{dy}{dk}-\frac{dx}{dk}\cdot\frac{dy}{dh}\right)=0. \quad \ldots \ldots \quad (9.)$$

Upon performing the operations here indicated upon the values of the coordinates in equations (2.) and (3.), the value of the quantity in brackets is found to be

$$1-e^{-\frac{2k}{R}}; \quad \ldots \ldots \quad (10.)$$

which is obviously independent of the time, and therefore fulfils the condition of continuity.

APPENDIX.

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On the Friction between a Wave and a Wave-shaped Solid.

Conceive that the trough between two consecutive crests of the trochoidal surface of a series of waves is occupied, for a breadth which may be denoted by z, by a solid body with a trochoidal surface, exactly fitting the wave-surface; that the solid body moves forward with a uniform velocity equal to that of the propagation of the waves, so as to continue always to fit the wave-surface, and that there is friction between the solid surface and the contiguous liquid particles, according to the law which experiment has shown to be at least approximately true, viz. varying as the surface of contact, and as the square of the velocity of sliding.

Conceive, further, that each particle of the liquid has that pressure applied to it which is required in order to keep its motion sensibly the same as if there were no friction; the solid body must of course be urged forwards by a pressure equal and opposite to the resultant of all the before-mentioned pressures.

The action, amongst the liquid particles, of pressures sufficient to overcome the friction, will disturb to a certain extent the motions of the liquid particles, and the figures of the surfaces of uniform pressure; but it will be assumed that those disturbances are small enough to be neglected, for the purposes of the present inquiry. The smallness of the pressures producing such disturbances, and consequently the smallness of those disturbances themselves, may be inferred from the fact, that the friction of a current of water over a surface of painted iron of a given area is equal to the weight of a layer of water covering the same area, and of a thickness which is only about ·0036 of the height due to the velocity of the current.

Those conditions having been assumed, let it now be proposed, *to find approximately the amount of resultant pressure required to overcome the friction between the wave and the wave-shaped solid.*

This problem is to be solved by finding the mechanical work expended in overcoming friction in an indefinitely small time dt, and dividing that work by the distance through which the solid moves in that time.

Taking, as before, as an independent variable the *phase* θ, being the angle which the tracing-arm $\overline{\text{CB}}=r$ (fig. 1) makes with a line pointing vertically downwards, the length of the elementary arc corresponding to an indefinitely small increment of phase $d\theta$ is

$$qd\theta,$$

where q is taken, for brevity's sake, to denote the normal AB.

The area of the corresponding element of the solid surface is

$$zqd\theta.$$

The velocity of sliding of the liquid particles over that elementary surface is

$$aq,$$

in which a, as before, denotes $\frac{d\theta}{dt}$, the angular velocity of the tracing-arm. Hence let ϱ denote the heaviness (or weight of unity of volume) of the liquid, and f its coefficient of friction when sliding over the given solid surface; the intensity of the friction per unit of area is

$$\frac{f\varrho a^2q^2}{2g}.$$

That friction has to be overcome, during the time dt, through the distance

$$aqdt=qd\theta.$$

Multiplying now together the elementary area, the intensity of the friction, and the distance through which it is overcome in the time dt, we find the following value for the work performed in that time in overcoming the friction at the given elementary surface,

$$zqd\theta\times\frac{f\varrho a^2q^2}{2g}\times qd\theta=\frac{f\varrho a^2}{2g}\cdot q^4zd\theta^2.$$

Now during the time dt, the solid advances through the distance

$$a\mathrm{R}dt=\mathrm{R}d\theta$$

(R, as before, being the radius of the rolling circle); and dividing the elementary portion of work expressed above by that distance, we find the following value for an elementary portion of the pressure required to overcome the friction,

$$d\mathrm{P}=\frac{f\varrho a^2}{2g}\cdot\frac{q^4z}{\mathrm{R}}\cdot d\theta. \qquad (1.)$$

The total pressure required to overcome the friction is found by integrating the preceding expression throughout an entire revolution, that is to say,

$$\mathrm{P}=\frac{f\varrho a^2z}{2g\mathrm{R}}\int_0^{2\pi}q^4d\theta. \qquad (2.)$$

To obtain this integral the following value of the square of the normal q or AB is to be substituted,

$$q^2=\mathrm{R}^2+r^2+2\mathrm{R}r.\cos\theta,$$

whence

$$\int_0^{2\pi}q^4d\theta=\mathrm{R}^4\int_0^{2\pi}\left(1+\frac{2r^2}{\mathrm{R}^2}+\frac{r^4}{\mathrm{R}^4}+4\left(1+\frac{r^2}{\mathrm{R}^2}\right)\frac{r}{\mathrm{R}}\cdot\cos\theta+4\,\frac{r^2}{\mathrm{R}^2}\cos^2\theta\right)d\theta=2\pi\mathrm{R}^4\left(1+4\frac{r^2}{\mathrm{R}^2}+\frac{r^4}{\mathrm{R}^4}\right),$$

and

$$\mathrm{P}=\frac{2\pi f\varrho a^2\mathrm{R}^3z}{2g}\cdot\left(1+4\frac{r^2}{\mathrm{R}^2}+\frac{r^4}{\mathrm{R}^4}\right). \qquad (3.)$$

The following modification of this expression is sometimes convenient:—

Let $\mathrm{V}=a\mathrm{R}$ denote the velocity of advance of the solid;

$\lambda=2\pi\mathrm{R}$, as before, its length, being the length of a wave;

$\sin\beta=\frac{r}{\mathrm{R}}$ the sine of the greatest angle made by a tangent to the trochoidal surface with the direction of advance; then

$$\mathrm{P}=\frac{f\varrho\mathrm{V}^2}{2g}\cdot\lambda z(1+4\sin^2\beta+\sin^4\beta\,*). \qquad (4.)$$

* This formula (neglecting $\sin^4\beta$ as unimportant in practice) has been used to calculate approximately the resistance of steam-vessels, and its results have been found to agree very closely with those of experiment, and have also been used since 1858 by Mr. James R. Napier and the author with complete success in practice, to calculate beforehand the engine-power required to propel proposed vessels at given speeds. The formula has been found to answer approximately, even when the lines of the vessel are not trochoidal, by putting for β

It is to be observed that the resistance P, as determined by the preceding investigation, being deduced from the amount of work performed against friction, includes not only the longitudinal components of the direct action of friction on each element of the surface of the solid, but the longitudinal components of the excess of the hydrostatic pressure against the front of the solid above that against its rear, which is the indirect effect of friction. The only quantities neglected are those arising from the disturbances of the figures of the surfaces of equal pressure, which quantities are assumed to be unimportant, for reasons already stated. The consideration of such quantities would introduce terms into the resistance varying as the fourth and higher powers of the velocity.

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NOTE, *added in October* 1862.

The investigation of Mr. STOKES (Camb. Trans. vol. viii.) proceeds to the second degree of approximation in shallow water, and to the third degree in water indefinitely deep. In the latter case he arrives at the result, that the crests of the waves rise higher above the level of still water than the troughs sink below that level, by a height agreeing with that stated in art. 14 of this paper, and that the profile of the waves is *approximately* trochoidal.

Mr. STOKES also arrives at the conclusion, that, when the disturbance is considerable compared with the length of a wave, there is combined with the orbital motion of each particle a *translation* which diminishes rapidly as the depth increases. No such translation has been found amongst the results of the investigation in the present paper; and hence it would appear that Mr. STOKES's results and mine represent two different possible modes of wave-motion*.

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the mean of the values of the greatest angle of obliquity for a series of water-lines. The method of using the formula in practice, and a Table showing comparisons of its results with those of experiment, were communicated to the British Association in 1861, and printed in the Civil Engineer and Architect's Journal for October of that year, and in part also in the 'Mechanics' Magazine,' 'The Artisan,' and 'The Engineer.' The ordinary value of the coefficient of friction f appears to be about ·0036 for water gliding over painted iron. The quantity $\lambda z(1+4\sin^2\beta+\sin^4\beta)$ corresponds to what is called, in the paper referred to, the *augmented surface.*

* NOTE *added in June* 1863.

The difference between the cases considered by Mr. STOKES and by me is the following:—In Mr. STOKES's investigation, the *molecular rotation* is null; that is to say,

$$\tfrac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right)=0\,;$$

while in my investigation it is constant in each layer, being the following function of k,

$$\tfrac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right)=\frac{ar_0^2e^{-\frac{2k}{R}}}{R^2-r_0^2e^{-\frac{2k}{R}}}. \quad \text{(11.)}$$

From this last equation it follows that

$$\frac{d}{dt}\left(\frac{dv}{dx}-\frac{du}{dy}\right)=0\,;$$

and therefore that the condition of continuity of pressure is verified.

The simplicity with which an exact result is obtained in the present paper, is entirely due to the following peculiarity:—Instead of taking for independent variables (besides the time) the *undisturbed* coordinates of a particle of liquid, there are taken two quantities, h and k, which are *functions* of those coordinates, of forms which are left indeterminate until the end of the investigation. h then proves to be identical with the undisturbed horizontal coordinate; but k proves to be a function of the undisturbed vertical coordinate for which there is no symbol in our present notation, being the root of the transcendental equation

$$k_0 - k - \frac{k_0^2}{2R} \cdot e^{-\frac{2k}{R}} = 0,$$

in which k_0 is the undisturbed vertical coordinate (see art. 13). Hence it is evident that, had k_0 instead of k been taken as the independent variable, the question of wave-motion considered in this paper could not have been solved except by a complex and tedious process of approximation.